Burgers velocity fields and the electromagnetic forcing in Schrödinger's interpolating dynamics

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Abstract

We explore a connection of the deterministically forced Burgers equation for local velocity fields with probabilistic solutions of the Schrödinger boundary data problem. An issue of deducing the most likely interpolating dynamics from the given initial and terminal probability density data is here investigated to give account of the perturbation by external electromagnetic fields. A suitable extension of the Hopf-Cole logarithmic transformation allows to deal with nonconservative drift fields and implies the validity of appropriately generalised heat equations, which completely determine the dynamics.

1 Burgers velocity fields and related transport processes

The Burgers equation with (usually without, [1, 2]) the forcing term $\vec{F}(\vec{x}, t)$:

$$\partial_t \vec{v} + (\vec{v}\nabla)\vec{v} = \nu\triangle\vec{v} + \vec{F}(\vec{x}, t) \tag{1}$$

and especially its statistically relevant $curl\,\vec{v} = 0$ solutions, recently have acquired a considerable popularity in the variety of physical contexts. They range from an astrophysical issue of the stratified large-scale distribution of matter in the early Universe, [3, 4, 5], through accoustic turbulence dealing with intense noise in compressible liquids and gases, [6], to primitive fluid turbulence modeling in terms of the statistics of Burgers shocks in the low viscosity regime (enhanced by random initial data), [3, 1, 7, 8], eventually ending with the analysis of a fully developed Burgers turbulence that is regarded as a result of random forcing (stirring) of respective velocity fields, [9, 10, 11]. It also pertains to the turbulence-without-pressure models, [12], description of directed polymers in a random potential, [11], random interface growth problem, [13], and fluctuations/dispersion in deterministic or random flows, [13, 14, 15, 16]. Last but not least, all that implies quite nontrivial mathematics (very prohibitive with respect to the widely spread formal white noise manipulations), [17, 18, 19].

Although Burgers velocity fields can be analysed on their own, frequently one needs a supplementary insight into the matter transport dynamics that is consistent with the chosen (Burgers) velocity field evolution. Then, the passive scalar (tracer or contaminant) advection-in-a-flow problem, [14, 11, 16] naturally appears through the parabolic dynamics:

$$\partial_t T + (\vec{v}\nabla)T = \nu\triangle T \tag{2}$$

Here, the Burgers velocity field defines the flow which amplifies the "passive scalar dispersion" (in analogy to the standard turbulent transport effects). This in turn implies the time-evolution of the concentration $T(\vec{x}, t)$ of the contaminant in the driving medium (incompressible fluid, $\nabla\vec{v} = 0$, but infinitely compressible gas as well). While looking for the stochastic implementation of the microscopic (molecular) dynamics, [21, 11, 16], it is assumed that the "diffusing scalar" (contaminant in the lore of early statistical turbulence models) obeys an Itô equation:

$$d\vec{X}(t) = \vec{v}(\vec{x}, t)dt + \sqrt{2\nu}d\vec{W}(t) \tag{3}$$

$$\vec{X}(0) = \vec{x}_0 \rightarrow \vec{X}(t) = \vec{x}$$

where the given forced Burgers velocity field is perturbed by the noise term representing a molecular diffusion. In the Itô representation of diffusion-type random variable $\vec{X}(t)$ one explicitly refers to the Wiener process $\sqrt{2\nu}\vec{W}(t)$, instead of the usually adopted formal white noise integral $\int_0^t \vec{\eta}(s)ds$, coming from the Langevin-type version of (3).

Under these premises, we cannot view equations (1)-(3) as completely independent (disjoint) problems: the velocity field $\vec{v}$ cannot be arbitrarily inferred from (1) or any other velocity-defining equation without verifying the *consistency* conditions, which would allow to associate (2) and (3) with a well defined random dynamics (stochastic process), and Markovian diffusion in particular, [22, 23]. In connection with the usage of Burgers velocity fields (with or without external forcing) which in (3) clearly are intended to replace the standard *forward drift* of the would-be-involved Markov diffusion process, we have not found in the literature any attempt to resolve apparent contradictions arising if (2) and/or (3) are defined by means of (1).

Moreover, rather obvious hesitation could have been observed in attempts to establish the most appropriate matter transport rule, if (1)-(3) are adopted. Depending on the particular phenomenological departure point, one either adopts the standard continuity equation, [3, 4], that is certainly valid to a high degree of accuracy in the low viscosity limit $\nu \downarrow 0$ of (1)-(3), but incorrect on mathematical grounds *if* there is a diffusion involved *and* simultaneously a solution of (1) stands for the respective *current* velocity of the flow: $\partial_t \rho(\vec{x}, t) = -\nabla[\vec{v}(\vec{x}, t)\rho(\vec{x}, t)]$. Alternatively, following the white noise calculus tradition telling that the stochastic integral $\vec{X}(t) = \int_0^t \vec{v}(\vec{X}(s), s)ds + \int_0^t \vec{\eta}(s)ds$ necessarily implies the Fokker-Planck equation, one adopts, [21]: $\partial_t \rho(\vec{x}, t) = \nu \triangle \rho(\vec{x}, t) - \nabla[\vec{v}(\vec{x}, t)\rho(\vec{x}, t)]$ which is clearly problematic in view of the classic Mc Kean's discussion of the propagation of chaos for the Burgers equation, [24, 25, 26] and the derivation of the stochastic "Burgers process" in this context: "the fun begins in trying to describe this Burgers motion as the path of a tagged molecule in an infinite bath of like molecules", [24].

Moreover, an issue of the necessary *correlation* (cf. [16], Chap.7.3, devoted to the turbulent transport and the related dispersion of contaminants) between the probabilistic Fokker-Planck dynamics of the diffusing tracer, and this of the passive tracer (contaminant) concentration (2), usually is left aside in the literature.

To put things on the solid ground, let us consider a Markovian diffusion process, which is characterised by the transition probability density (generally inhomogeneous in space and time law of random displacements) $p(\vec{y}, s, \vec{x}, t)$, $0 \le s < t \le T$, and the probability density $\rho(\vec{x}, t)$ of its random variable $\vec{X}(t)$, $0 \le t \le T$. The process is completely determined by these data. For clarity of discussion, we do not impose any spatial boundary restrictions, nor fix any concrete limiting value of T

which, in principle, can be moved to infinity.

The conditions valid for any $\epsilon > 0$:

(a) there holds $lim_{t\downarrow s}\frac{1}{t-s}\int_{|\vec{y}-\vec{x}|>\epsilon}p(\vec{y},s,\vec{x},t)d^3x = 0$,

(b) there exists a (forward) drift $\vec{b}(\vec{x},s) = lim_{t\downarrow s}\frac{1}{t-s}\int_{|\vec{y}-\vec{x}|\leq\epsilon}(\vec{y}-\vec{x})p(\vec{x},s,\vec{y},t)d^3y$,

(c) there exists a diffusion function (in our case it is simply a diffusion coefficient ν) $a(\vec{x},s) = lim_{t\downarrow s}\frac{1}{t-s}\int_{|\vec{y}-\vec{x}|\leq\epsilon}(\vec{y}-\vec{x})^2p(\vec{x},s,\vec{y},t)d^3y$,

are conventionally interpreted to define a diffusion process, [23, 22]. Under suitable restrictions (boundedness of involved functions, their continuous differentiability) the function:

$$g(\vec{x},s) = E\{g(\vec{X}(T))|\vec{X}(s) = \vec{x}; s \leq T\} = \int p(\vec{x},s,\vec{y},T)g(\vec{y},T)d^3y \qquad (4)$$

satisfies the backward diffusion equation (notice two sign changes in comparison with (2))

$$-\partial_s g(\vec{x},s) = \nu\triangle g(\vec{x},s) + [\vec{b}(\vec{x},s)\nabla]g(\vec{x},s) \ . \qquad (5)$$

Let us point out that the validity of (5) is known to be a *necessary* condition for the existence of a Markov diffusion process, whose probability density $\rho(\vec{x},t)$ is to obey the Fokker-Planck equation (the forward drift $\vec{b}(\vec{x},t)$ replaces the previously utilized Burgers velocity $\vec{v}(\vec{x},t)$):

$$\partial_t\rho(\vec{x},t) = \nu\triangle\rho(\vec{x},t) - \nabla[\vec{b}(\vec{x},t)\rho(\vec{x},t)] \qquad (6)$$

The case of particular interest, in the traditional nonequilibrium statistical physics literature, appears when $p(\vec{y},s,\vec{x},t)$ is a *fundamental solution* of (5) with respect to variables $\vec{y},s$, [27, 22, 23], see however [28] for an analysis of alternative situations. Then, the transition probability density satisfies *also* the second Kolmogorov (e.g. the Fokker-Planck) equation in the remaining $\vec{x},t$ pair of variables. Let us emphasize that these two equations form an adjoint pair of partial differential equations, referring to the slightly counterintuitive for physicists, though transparent for mathematicians, [30, 31, 32, 33, 34, 35], issue of time reversal of diffusions.

After adjusting (3) to the present context, $\vec{X}(t) = \int_0^t \vec{b}(\vec{X}(s),s)\,ds + \sqrt{2\nu}\vec{W}(t)$ we can utilize the standard rules of the Itô stochastic calculus, [36, 33, 34, 35], to realise that for any smooth function $f(\vec{x},t)$ of the random variable $\vec{X}(t)$ the conditional expectation value:

$$lim_{\triangle t\downarrow 0}\frac{1}{\triangle t}[\int p(\vec{x},t,\vec{y},t+\triangle t)f(\vec{y},t+\triangle t)d^3y - f(\vec{x},t)] = (D_+f)(\vec{X}(t),t) = \qquad (7)$$

$$= (\partial_t + \vec{b}\nabla + \nu\triangle)f(\vec{x},t) \ ,$$

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where $\vec{X}(t) = \vec{x}$, determines the forward drift $\vec{b}(\vec{x}, t)$ (if we set components of $\vec{X}$ instead of f) and allows to introduce the local field of (forward) accelerations associated with the diffusion process, which we *constrain* by demanding (see e.g. Refs. [33, 36, 34, 35] for prototypes of such dynamical constraints):

$$(D_+^2 \vec{X})(t) = (D_+ \vec{b})(\vec{X}(t), t) = (\partial_t \vec{b} + (\vec{b}\nabla)\vec{b} + \nu \triangle \vec{b})(\vec{x}, t) = \vec{F}(\vec{x}, t) \qquad (8)$$

where $\vec{X}(t) = \vec{x}$ and, at the moment arbitrary, function $\vec{F}(\vec{x}, t)$ may be interpreted as an external forcing applied to the diffusing system, [29].

In particular, if we assume that drifts remain gradient fields, $curl\, \vec{b} = 0$, under the forcing, then those that are allowed by the prescribed choice of $\vec{F}(\vec{x}, t)$ *must* fulfill the compatibility condition (notice the conspicuous absence of the standard Newtonian minus sign in this analogue of the second Newton law, see e.g. [37, 34, 29])

$$\vec{F}(\vec{x}, t) = \nabla \Omega(\vec{x}, t) \qquad (9)$$

$$\Omega(\vec{x}, t) = 2\nu[\partial_t \Phi + \frac{1}{2}(\frac{b^2}{2\nu} + \nabla b)]$$

which establishes the Girsanov-type martingale connection of the forward drift $\vec{b}(\vec{x}, t) = 2\nu \nabla \Phi(\vec{x}, t)$ with the Feynman-Kac, cf. [29, 28], potential $\Omega(\vec{x}, t)$ of the chosen external force field.

One of the distinctive features of Markovian diffusion processes with the positive density $\rho(\vec{x}, t)$ is that the notion of the *backward* transition probability density $p_*(\vec{y}, s, \vec{x}, t)$ can be consistently introduced on each finite time interval, say $0 \le s < t \le T$:

$$\rho(\vec{x}, t)p_*(\vec{y}, s, \vec{x}, t) = p(\vec{y}, s, \vec{x}, t)\rho(\vec{y}, s) \qquad (10)$$

so that $\int \rho(\vec{y}, s)p(\vec{y}, s, \vec{x}, t)d^3y = \rho(\vec{x}, t)$ and $\rho(\vec{y}, s) = \int p_*(\vec{y}, s, \vec{x}, t)\rho(\vec{x}, t)d^3x$. This allows to define the backward derivative of the process in the conditional mean (cf. [29, 38, 39] for a discussion of these concepts in case of the most traditional Brownian motion and Smoluchowski-type diffusion processes)

$$lim_{\triangle t \downarrow 0} \frac{1}{\triangle t}[\vec{x} - \int p_*(\vec{y}, t - \triangle t, \vec{x}, t)\vec{y}d^3y] = (D_- \vec{X})(t) = \vec{b}_*(\vec{X}(t), t) \qquad (11)$$

$$(D_- f)(\vec{X}(t), t) = (\partial_t + \vec{b}_*\nabla - \nu\triangle)f(\vec{X}(t), t)$$

Accordingly, the backward version of the acceleration field reads

$$(D_-^2 \vec{X})(t) = (D_+^2 \vec{X})(t) = \vec{F}(\vec{X}(t), t) \qquad (12)$$

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where in view of $\vec{b}_* = \vec{b} - 2\nu\nabla ln\rho$ we have explicitly fulfilled the forced Burgers equation (cf. (1)):

$$\partial_t \vec{b}_* + (\vec{b}_*\nabla)\vec{b}_* - \nu\triangle\vec{b}_* = \vec{F} \tag{13}$$

and, [33, 34, 29], under the gradient-drift field assumption, $curl\,\vec{b}_* = 0$, we deal with $\vec{F}(\vec{x}, t)$ given by (9) again. A notable consequence of the involved backward Itô calculus is that the Fokker-Planck equation (6) can be transformed to an *equivalent* form of:

$$\partial_t\rho(\vec{x}, t) = -\nu\triangle\rho(\vec{x}, t) - \nabla[\vec{b}_*(\vec{x}, t)\rho(\vec{x}, t)] \tag{14}$$

with the very same initial (Cauchy) data $\rho_0(\vec{x}) = \rho(\vec{x}, 0)$ as in (6).

At this point let us recall that equations (5) and (6) form a natural adjoint pair of equations for the diffusion process in the time interval $[0, T]$. Clearly, an adjoint of (14) reads:

$$\partial_s f(\vec{x}, s) = \nu\triangle f(\vec{x}, s) - [\vec{b}_*(\vec{x}, s)\nabla]f(\vec{x}, s) \tag{15}$$

where:

$$f(\vec{x}, s) = \int p_*(\vec{y}, 0, \vec{x}, s)f(\vec{y}, 0)d^3y \ , \tag{16}$$

to be compared with (4),(5) and the previously mentioned passive scalar dynamics (2), see e.g. also [21]. Here, manifestly, the time evolution of the backward drift is governed by the Burgers equation, and the diffusion equation (15) is correlated (via the definition (10)) with the probability density evolution rule (14).

This pair *only* can be consistently utilized if the diffusion proces is to be driven by forced (or unforced) Burgers velocity fields.

On the other hand, the study of diffusion driven by the Burgers flow may begin from first solving the Burgers equation (12) for a chosen external force field, next specifying the probability density (14), eventually ending with the corresponding "passive contaminant" concentration dynamics (15), (16). All that remains in a perfect agreement with the heuristic discussion of the concentration dynamics given in Ref. [16], Chap. 7.3. where the "backward dispersion" problem with "time running backwards" was found necessary to *predict* the concentration.

Let us notice that the familiar logarithmic Hopf-Cole transformation, [2, 43], of the Burgers equation into the generalised diffusion equation (yielding explicit solutions in the unforced case) has received a generalisation in the framework of the so called Schrödinger interpolation problem, [34, 35, 28, 29, 39, 40], see also [45, 42]. In its recent reformulation, [28], the solution in terms of the interpolating Markovian diffusion process is found to rely on the adjoint pairs of parabolic equations, like e.g. (5), (6) or (14), (15). In case of gradient drift fields the process can be determined by checking (this imposes limitations on the admissible force field potential) whether

the Feynman-Kac kernel

$$k(\vec{y}, s, \vec{x}, t) = \int exp[-\int_s^t c(\vec{\omega}(\tau), \tau)d\tau]d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\omega) \tag{17}$$

is positive and continuous in the open space-time area of interest (then, additional limitations on the path measure need to be introduced, [29, 41]), and whether it gives rise to positive solutions of the adjoint pair of generalised heat equations:

$$\partial_t u(\vec{x}, t) = \nu \triangle u(\vec{x}, t) - c(\vec{x}, t)u(\vec{x}, t) \tag{18}$$

$$\partial_t v(\vec{x}, t) = -\nu \triangle v(\vec{x}, t) + c(\vec{x}, t)v(\vec{x}, t)$$

Here, a function $c(\vec{x}, t) = \frac{1}{2\nu}\Omega(\vec{x}, t)$ is restricted only by the positivity and continuity demand for the kernel (17), see e.g. [42, 28]. In the above, $d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\omega)$ is the conditional Wiener measure over sample paths of the standard Brownian motion.

Solutions of (18), upon suitable normalisation give rise to the Markovian diffusion process with the factorised probability density $\rho(\vec{x}, t) = u(\vec{x}, t)v(\vec{x}, t)$ wich interpolates between the boundary density data $\rho(\vec{x}, 0)$ and $\rho(\vec{x}, T)$, with the forward and backward drifts of the process defined as follows:

$$\vec{b}(\vec{x}, t) = 2\nu \frac{\nabla v(\vec{x}, t)}{v(\vec{x}, t)} \tag{19}$$

$$\vec{b}_*(\vec{x}, t) = -2\nu \frac{\nabla u(\vec{x}, t)}{u(\vec{x}, t)}$$

in the prescribed time interval $[0, T]$. These formulas imply that the compatibility condition (9), connecting the drifts with the a priori chosen function $c(\vec{x}, t) = \frac{1}{2\nu}\Omega(\vec{x}, t)$, can be easily established.

The transition probability density of this process reads:

$$p(\vec{y}, s, \vec{x}, t) = k(\vec{y}, s, \vec{x}, t)\frac{v(\vec{x}, s)}{v(\vec{y}, t)} \quad . \tag{20}$$

while the corresponding (since $\rho(\vec{x}, t)$ is given) transition probability density, (10), of the backward process has the form:

$$p_*(\vec{y}, s, \vec{x}, t) = k(\vec{y}, s, \vec{x}, t)\frac{u(\vec{y}, s)}{u(\vec{x}, t)} \quad . \tag{21}$$

Obviously, [28, 34], in the time interval $0 \le s < t \le T$ there holds:

$$u(\vec{x}, t) = \int u_0(\vec{y})k(\vec{y}, s, \vec{x}, t)d^3y \tag{22}$$

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$$v(\vec{y}, s) = \int k(\vec{y}, s, \vec{x}, T) v_T(\vec{x}) d^3x \ .$$

By defining $\Phi_* = log\, u$, we immediately recover the traditional form of the Hopf-Cole transformation for Burgers velocity fields: $\vec{b}_* = -2\nu\nabla\Phi_*$. In the special case of the standard free Brownian motion, there holds $\vec{b}(\vec{x}, t) = 0$ while $\vec{b}_*(\vec{x}, t) = -2\nu\nabla log\, \rho(\vec{x}, t)$.

2 The problem of electromagnetic forcing in the Schrödinger interpolating dynamics

It turns out the the crucial point of our previous discussion lies in a *proper* choice of the strictly positive and continuous, in an open space-time area, function $k(\vec{y}, s, \vec{x}, t)$ which, if we wish to construct a Markov process, has to satisfy the Chapman-Kolmogorov (semigroup composition) equation. It has led us to consider a pair of adjoint partial differential equations, (18), as an alternative to either (5), (6) or (14), (15).

In the quantally oriented literature dealing with Schrödinger operators and their spectral properties, [46, 47, 41], the potential $c(x, t)$ in (17), (18) is usually assumed to be a continuous and bounded from below function, but these restrictions can be substantially relaxed (unbounded functions are allowed in principle) if we wish to consider general Markovian diffusion processes and disregard an issue of the bound state spectrum and this of the ground state of the (selfadjoint) semigroup generator, [27, 22]. Actually, what we need is merely that properties of $c(\vec{x}, t)$ allow for the kernel k, (17), which is positive and continuous. Taking for granted that suitable conditions are fulfilled, [46, 28], we can immediately associate with equations (18) an integral kernel of the time-dependent semigroup (the exponential operator should be understood as time-ordered expression, since in general $H(\tau)$ may not commute with $H(\tau')$ for $\tau \neq \tau'$):

$$k(\vec{y}, s, \vec{x}, t) = [exp(-\int_s^t H(\tau)d\tau)](\vec{y}, \vec{x}) \tag{23}$$

where $H(\tau) = -\nu\triangle + c(\tau)$ is the pertinent semigroup generator. Then, by the Feynman-Kac formula, [42], we get an expression (17) for the kernel, which in turn yields (19)-(22), see e.g. [28]. As mentioned before, (20) combined with (17) sets a Girsanov probabilistic (martingale) connection between the Wiener measure corresponding to the standard Brownian motion with $\vec{b}(\vec{x}, t) = 0$, and this appropriate for the diffusion process with a nonvanishing drift $\vec{b}(\vec{x}, t)$, $curl\, \vec{b} = 0$.

The above formalism is known, [29], to encompass the standard Smoluchowski-type diffusions in conservative force fields.

Strikingly, an investigation of electromagnetically forced diffusions has not been much pursued in the literature, although an issue of deriving the Smoluchowski-Kramers equation (and possibly its large friction limit) from the Langevin-type equation for the charged Brownian particle in the general electromagnetic field has been relegated in Ref. [48], Chap. 6.1 to the status of the innocent-looking exercise (sic !). On the other hand, the diffusion of realistic charges in dilute ionic solutions creates a number of additional difficulties due to the apparent Hall mobility in terms of mean currents induced by the electric field (once assumed to act upon the system), see e.g. [49, 50] and [51].

In connection with the electromagnetic forcing of diffusing charges, the gradient field assumption imposes a severe limitation if we account for typical (nonzero circulation) features of the classical motion due to the Lorentz force, with or without the random perturbation component. The purely electric forcing is simpler to handle, since it has a definite gradient field realisation, see e.g. [52] for a recent discussion of related issues. The major obstacle with respect to our previous (Section 1) discussion is that, if we wish to regard either the force $\vec{F}$, (8), (12), or drifts $\vec{b}$, $\vec{b}_*$ to have an electromagnetic origin, then necessarily we need to pass from conservative to non-conservative fields. This subject matter has not been significantly exploited so far in the nonequilibrium statistical physics literature.

Interestingly, in the framework of the Onsager-Machlup approach towards an identification of most probable paths which one can associate with the underlying diffusion process, [53, 54, 55], the non-conservative model system has been investigated, [56]. Namely, an effectively two-dimensional Brownian motion was analyzed, whose three-dimensional forward drift $\vec{b}(\vec{x})$, $b_3 = 0$ in view of $\partial_x b_1 \neq \partial_y b_2$ has $curl\,\vec{b} \neq 0$. Then, by the standard variational argument with respect to the Wiener-Onsager-Machlup action, [54, 56],

$$I\{L(\dot{\vec{x}}, \vec{x}, t); t_1, t_2\} = \frac{1}{2\nu} \int_{t_1}^{t_2} \{\frac{1}{2}[\dot{\vec{x}} - \vec{b}(\vec{x}, t)]^2 + \nu \nabla \vec{b}(\vec{x}, t)\} dt \ , \qquad (24)$$

the most probable trajectory, about which major contributions from (weighted) Brownian paths are concentrated, was found to be a solution of the Euler-Lagrange equations, which are formally identical to the equations of motion

$$\ddot{\vec{q}}_{cl} = \vec{E} + \dot{\vec{q}}_{cl} \times \vec{B} \qquad (25)$$

of a classical particle of unit mass and unit charge moving in an electric field $\vec{E}$ and the magnetic field $\vec{B}$. The electric field (to be compared with Eq. (9)) is given by:

$$\vec{E} = -\nabla \Phi \qquad (26)$$

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$$\Phi = -\frac{1}{2}(\vec{b}^2 + 2\nu\nabla\vec{b})$$

while the magnetic field has the only nonvanishing component in the z-direction of R^3:

$$\vec{B} = curl\,\vec{b} = \{0, 0, \partial_x b_2 - \partial_y b_1\} \quad , \tag{27}$$

Clearly, $\vec{B} = curl\vec{A}$ where $\vec{A} \dot{=} \vec{b}$ is the electromagnetic vector potential. The simplest example is a notorious constant magnetic field defined by $b_1(\vec{x}) = -\frac{B}{2}x_2$, $b_2(\vec{x}) = \frac{B}{2}x_1$.

One immediately realizes that the Fokker-Planck equation in this case is incompatible with traditional intuitions underlying the Smoluchowski-drift identification: the forward drift is not proportional to an external force, but to an electromagnetic potential. Nevertheless, the variational information drawn from the Onsager-Machlup Lagrangian involves the Lorentz force-driven trajectory. Hence, some principal effects of the electromagnetic forcing are present in the diffusing system, whose drifts display an "unphysical" (gauge dependent) form.

On the other hand, if we accept this "unphysical" random motion to yield the representation $d\vec{X}(t) = \vec{A}(\vec{X}(t), t)dt + \sqrt{2\nu}d\vec{W}(t)$, supplemented by the corresponding pair (5), (6) of adjoint diffusion equations with $\vec{A}(\vec{x}, t)$ replacing $\vec{b}(\vec{x}, t)$, then standard rules of the Itô stochastic calculus, (8), tell us that

$$(D_+^2\vec{X})(t) = \partial_t\vec{A} + (\vec{A}\nabla)\vec{A} + \nu\triangle\vec{A} = -\frac{B^2}{2}\{x_1, x_2, 0\} = -\vec{E}(\vec{x}) \tag{28}$$

where $\vec{E}(\vec{x}) = \frac{B^2}{2}\{x_1, x_2, 0\}$, if calculated from (26).

We thus arrive at the purely electric forcing with reversed sign (if compared with that coming from the Onsager-Machlup argument, (26)) and somewhat surprisingly, there is no impact of the previously discussed magnetic motion on the level of dynamical constraints (8), (13). The adopted recipe is thus incapable of producing the magnetically forced diffusion process that conforms with arguments of Section 1.

In below, we shall follow more abstract route, based on invoking the Feynman-Kac kernel idea (23), [28]. This approach has a clear advantage of elucidating the generic issues that hamper attempts to describe electromagnetically forced diffusion processes.

Usually, the selfadjoint semigroup generators attract the attention of physicists in connection with the Feynman-Kac formula. A typical route towards incorporating electromagnetism comes from quantal motivations via the minimal electromagnetic coupling recipe which preserves the selfadjointness of the generator (Hamiltonian of the system). As such, it constitutes a part of the general theory of Schrödinger operators. A rigorous study of operators of the form $-\triangle + V$ has become a well

developed mathematical discipline, [46]. The study of Schrödinger operators with magnetic fields, typically of the form $-(\nabla - i\vec{A})^2 + V$, is less advanced, although specialised chapters on the magnetic field issue can be found in monographs devoted to functional integration methods, [46, 57], mostly in reference to seminal papers [58, 59].

From the mathematical point of view, it is desirable to deal with magnetic fields that go to zero at infinity, which is certainly acceptable on physical grounds as well. The constant magnetic field (see e.g. our previous considerations) does not meet this requirement, and its notorious usage in the literature makes us (at the moment) to decline the asymptotic assumption and inevitably fall into a number of serious complications.

One obvious obstacle can be seen immediately by taking advantage of the existing results, [58]. Namely, an explicit expression for the Feynman-Kac kernel in a constant magnetic field, introduced through the the minimal electromagnetic coupling assumption $H(\vec{A}) = -\frac{1}{2}(\nabla - i\vec{A})^2$, is available (up to irrelevant dimensional constants):

$$exp[-tH(\vec{A})](\vec{x}, \vec{y}) = [\frac{B}{4\pi sinh(\frac{1}{2}Bt)}](\frac{1}{2\pi t})^{1/2} \tag{29}$$

$$\times exp\{-\frac{1}{2t}(x_3 - y_3)^2 - \frac{B}{4}coth(\frac{B}{2}t)[(x_2 - y_2)^2 + (x_1 - y_1)^2] - i\frac{B}{2}(x_1 y_2 - x_2 y_1)\} \ .$$

Clearly, it is *not* real (hence *non-positive* and directly at variance with the major demand in the Schrödinger interpolation problem, as outlined in Section 1), except for directions $\vec{y}$ that are parallel to a chosen $\vec{x}$.

Consequently, a bulk of the well developed mathematical theory is of no use for our purposes and new techniques must be developed for a consistent description of the electromagnetically forced diffusion processes along the lines of Section 1, i.e. within the framework of Schrödinger's interpolation problem.

3   Forcing via Feynman-Kac semigroups

The conditional Wiener measure $d\mu^{(\vec{y},s)}_{(\vec{x},t)}(\vec{\omega})$, appearing in the Feynman-Kac kernel definition (17), if unweighted (set $c(\vec{\omega}(\tau), \tau) = 0$) gives rise to the familiar heat kernel. This, in turn, induces the Wiener measure P_W of the set of all sample paths, which originate from $\vec{y}$ at time s and terminate (can be located) in the Borel set $A \in R^3$ after time $t - s$: $P_W[A] = \int_A d^3x \int d\mu^{(\vec{y},s)}_{(\vec{x},t)}(\vec{\omega}) = \int_A d\mu$ where, for simplicity of notation, the $(\vec{y}, t - s)$ labels are omitted and $\mu^{(\vec{y},s)}_{(\vec{x},t)}$ stands for the heat kernel.

Having defined an Itô diffusion $\vec{X}(t) = \int_0^t \vec{b}(\vec{x}, u)du + \sqrt{2\nu}\vec{W}(t)$ we are interested in the analogous path measure: $P_{\vec{X}}[A] = \int_A dx \int d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\vec{\omega}_{\vec{X}}) = \int_A d\mu(\vec{X})$.

Under suitable (stochastic, [29]) integrability conditions imposed on the forward drift, we have granted the absolute continuity $P_X \ll P_W$ of measures, which implies the existence of a strictly positive Radon-Nikodym density. Its canonical Cameron-Martin-Girsanov form, [46, 29], reads:

$$\frac{d\mu(\vec{X})}{d\mu}(\vec{y}, s, \vec{x}, t) = exp\frac{1}{2\nu}[\int_s^t \vec{b}(\vec{X}(u), u)d\vec{X}(u) - \frac{1}{2}\int_s^t [\vec{b}(\vec{X}(u), u)]^2 du] \ . \quad (30)$$

If we assume that drifts are gradient fields, $curl\,\vec{b} = 0$, then the Itô formula allows to reduce, otherwise troublesome, stochastic integration in the exponent of (30), [46, 57], to ordinary Lebesgue integrals:

$$\frac{1}{2\nu}\int_s^t \vec{b}(\vec{X}(u), u)d\vec{X}(u) = \Phi(\vec{X}(t), t) - \Phi(\vec{X}(s), s) - \int_s^t du\, [\partial_t \Phi + \frac{1}{2}\nabla\vec{b}](\vec{X}(u), u) \ .$$
$$(31)$$

After inserting (31) to (30) and next integrating with respect to the conditional Wiener measure, on account of (9) we arrive at the standard form of the Feynman-Kac kernel (17). Notice that (31) establishes a probabilistic basis for logarithmic transformations (19) of forward and backward drifts: $b = 2\nu\nabla log\, v = 2\nu\nabla\Phi$, $b_* = -2\nu\nabla log\, u = -2\nu\nabla\Phi_*$. The forward version is commonly used in connection with the transformation of the Fokker-Planck equation into the generalised heat equation, [37, 44, 29]. The backward version is the Hopf-Cole transformation, mentioned in Section 1, used to map the Burgers equation into the very same generalised heat equation as in the previous case, [2, 45].

However, presently we are interested in non-conservative drift fields, $curl\,\vec{b} \neq 0$, and in that case the stochastic integral in (30) is the major source of computational difficulties, [46, 57, 33], for nontrivial vector potential field configurations. It explains the virtual absence of magnetically forced diffusion problems in the nonequilibrium statistical physics literature.

At this point, some steps of the analysis performed in Ref. [63] in the context of the "Euclidean quantum mechanics", cf. also [35], are extremely useful. Let us emphasize that electromagnetic fields we utilize, are always meant to be ordinary Maxwell fields with *no* Euclidean connotations (see e.g. Chap.9 of Ref. [57] for the Euclidean version of Maxwell theory).

Let us consider a gradient drift-field diffusion problem according to Section 1, with (17), (31) involved and thus an adjoint pair (18) of parabolic equations completely defining the Markovian diffusion process. Furthermore, let $\vec{A}(\vec{x})$ be the time-independent vector potential for the Maxwellian magnetic field $\vec{B} = curl\,\vec{A}$.

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We pass from the gradient realisation of drifts to the new one, generalizing (19), for which the following decomposition into the gradient and nonconservative part is valid:

$$\vec{b}(\vec{x}, t) = 2\nu \, log \, \Phi(\vec{x}, t) - \vec{A}(\vec{x}) \ , \tag{32}$$

We denote $\theta(\vec{x}, t) \doteq exp \, [\Phi(\vec{x}, t)]$ and admit that (32) is a forward drift of an Itô diffusion process with a stochastic differential $d\vec{X}(t) = [2\nu \frac{\nabla\theta}{\theta} - \vec{A}]dt + \sqrt{2\nu}d\vec{W}(t)$. On purely formal grounds, we deal here with an example of the Cameron-Martin-Girsanov transformation of the forward drift of a given Markovian diffusion process and we are entitled to ask for a corresponding measure transformation, (30).

To this end, let us furthermore *assume* that $\theta(\vec{x}, t) = \theta$ solves a partial differential equation

$$\partial_t \theta = -\nu[\nabla - \frac{1}{2\nu}\vec{A}(\vec{x})]^2\theta + c(\vec{x}, t)\theta \tag{33}$$

with the notation $c(\vec{x}, t) = \frac{1}{2\nu}\Omega(\vec{x}, t)$ patterned after (9). Then, by using the Itô calculus and (32), (33) on the way, see e.g. Ref. [63], we can rewrite (30) as follows:

$$\frac{d\mu(\vec{X})}{d\mu}(\vec{y}, s, \vec{x}, t) = \tag{34}$$

$$= exp \, \frac{1}{2\nu}[\int_s^t (2\nu\frac{\nabla\theta}{\theta} - \vec{A})(\vec{X}(u), u)d\vec{X}(u) - \frac{1}{2}\int_s^t (2\nu\frac{\nabla\theta}{\theta} - \vec{A})^2(\vec{X}(u), u) \, du]$$

$$= \frac{\theta(\vec{X}(t), t)}{\theta(\vec{X}(s), s)} exp[-\frac{1}{2\nu}\int_s^t [\vec{A}(u)d\vec{X}(u) \, + \, \nu(\nabla\vec{A})(\vec{X}(u))du \, + \, \Omega(\vec{X}(u), u)du]] \ ,$$

where $\vec{X}(s) = \vec{y}, \vec{X}(t) = \vec{x}$.

More significant observation is that the Radon-Nikodym density (34), if integrated with respect to the conditional Wiener measure, gives rise to the Feynman-Kac kernel (23) of the *non-selfadjoint* semigroup (suitable integrability conditions need to be respected here as well, [63]), with the generator $H_{\vec{A}} = -\nu[\nabla - \frac{1}{2\nu}\vec{A}(\vec{x})]^2 + c(\vec{x}, t)$ defined by the right-hand-side of (33):

$$\partial_t \theta(\vec{x}, t) = H_{\vec{A}}\theta(\vec{x}, t) =$$

$$= [-\nu\triangle + \vec{A}(\vec{x})\nabla + \frac{1}{2}(\nabla\vec{A}(\vec{x})) - \frac{1}{4\nu}[\vec{A}(\vec{x})]^2 + c(\vec{x}, t)]\theta(\vec{x}, t) \tag{35}$$

$$= -\nu\triangle\theta(\vec{x}, t) + \vec{A}(\vec{x})\nabla\theta(\vec{x}, t) + c_{\vec{A}}(\vec{x}, t)\theta(\vec{x}, t) \ .$$

Here:

$$c_A(\vec{x}, t) = c(\vec{x}, t) + \frac{1}{2}(\nabla\vec{A})(\vec{x}) - \frac{1}{4\nu}[\vec{A}(\vec{x})]^2 \ . \tag{36}$$

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An adjoint parabolic partner of (35) reads:

$$\partial_t \theta_* = -H_{\vec{A}}^* \theta_* = \nu \triangle \theta_* + \nabla[\vec{A}(\vec{x})\theta_*] - c_A(\vec{x}, t)\theta_* = \qquad (37)$$

$$= \nu[\nabla + \frac{1}{2\nu}\vec{A}(\vec{x})]^2 \theta_* - c(\vec{x}, t)\theta_* \ .$$

Consequently, our assumptions (32), (33) involve a generalization of the adjoint parabolic system (18) to a new adjoint one comprising (33), (37). Obviously, the original form of (18) is immediately restored by setting $\vec{A} = \vec{0}$, and executing obvious replacements $\theta_* \to u$, $\theta \to v$.

Let us emphasize again, that in contrast to Ref. [63], where the non-Hermitean generator $2\nu H_{\vec{A}}$, (33), has been introduced as "the Euclidean version of the Hamiltonian" $H = -2\nu^2(\nabla - \frac{i}{2\nu}\vec{A})^2 + \Omega$, our electromagnetic fields stand for solutions of the usual Maxwell equations and *are not* Euclidean at all.

As long as the coefficient functions (both additive and multiplicative) of the adjoint parabolic system (35), (37) are not specified, we remain within a general theory of positive solutions for parabolic equations with unbounded coefficients (of particular importance, if we do not impose any asymptotic fall off restrictions), [27, 60, 61, 62]. The fundamental solutions, if their existence can be granted, usually live on space-time strips, and generally do not admit unbounded time intervals. We shall disregard these issues at the moment, and assume the existence of fundamental solutions without any reservations.

By exploiting the rules of functional (Malliavin, variational) calculus, under an assumption that we deal with a diffusion (in fact, Bernstein) process associated with an adjoint pair (35), (36), it has been shown in Ref. [63] that *if* the forward conditonal derivatives of the process exist, then $(D_+\vec{X})(t) = 2\nu\frac{\nabla\theta}{\theta} - \vec{A} = \vec{b}(\vec{x}, t)$, (32) and:

$$(D_+^2\vec{X})(t) = (D_+\vec{X})(t) \times curl\,\vec{A}(\vec{x}) + \nabla\Omega(\vec{x}, t) + \nu curl\,(curl\,\vec{A}(\vec{x})) \qquad (38)$$

where $\vec{X}(0) = 0$, $\vec{X}(t) = \vec{x}$, $\times$ denotes the vector product in R^3 and $2\nu c = \Omega$.

Since $\vec{B} = curl\,\vec{A} = \mu_0\vec{H}$, we identify in the above the standard Maxwell equation for $curl\,\vec{H}$ comprising magnetic effects of electric currents in the system: $curl\,\vec{B} = \mu_0[\dot{\vec{D}} + \sigma_0\vec{E} + \vec{J}_{ext}]$ where $\vec{D} = \epsilon_0\vec{E}$ while $\vec{J}_{ext}$ represents external electric currents. In case of $\vec{E} = \vec{0}$, the external currents only would be relevant. A demand $curl\,curl\,\vec{A} = \nabla(\nabla\vec{A}) - \triangle\vec{A} = 0$ corresponds to a total absence of such currents, and the Coulomb gauge choice $\nabla\vec{A} = 0$ would leave us with harmonic functions $\vec{A}(\vec{x})$.

Consequently, a correct expression for the magnetically implemented Lorentz force has appeared on the right-hand-side of the forward acceleration formula (38),

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with the forward drift (32) replacing the classical particle velocity $\dot{\vec{q}}$ of the classical formula (25).

The above discussion implicitly involves quite sophisticated mathematics, hence it is instructive to see that we can bypass the apparent complications by directly invoking the universal definitions (7) and (11) of conditional expectation values, that are based on exploitation of the Itô formula only. Obviously, under an assumption that the Markovian diffusion process with well defined transition probability densities $p(\vec{y}, s, \vec{x}, t)$ and $p_*(\vec{y}, s, \vec{x}, t)$, does exist.

We shall utilize an obvious generalization of canonical definitions (19) of both forward and backward drifts of the diffusion process defined by the adjoint parabolic pair (18), as suggested by (32) with $\vec{A} = \vec{A}(\vec{x})$:

$$\vec{b} = 2\nu\frac{\nabla\theta}{\theta} - \vec{A} \ , \ \ \vec{b}_* = -2\nu\frac{\nabla\theta_*}{\theta_*} - \vec{A} \ . \tag{39}$$

We also demand that the corresponding adjoint equations (35), (37) *are* solved by θ and θ_* respectively.

Taking for granted that identities $(D_+\vec{X})(t) = \vec{b}(\vec{x}, t)$, $\vec{X}(t) = \vec{x}$ and $(D_-\vec{X})(t) = \vec{b}_*(\vec{x}, t)$ hold true, we can easily evaluate the forward and backward accelerations (substitute (39), and exploit the equations (35), (37)):

$$(D_+\vec{b})(\vec{X}(t), t) = \partial_t\vec{b} + (\vec{b}\nabla)\vec{b} + \nu\triangle\vec{b} = \tag{40}$$

$$= \vec{b} \times \vec{B} + \nu\,curl\,\vec{B} + \nabla\,\Omega$$

and

$$(D_-\vec{b}_*)(\vec{X}(t), t) = \partial_t\vec{b}_* + (\vec{b}_*\nabla)\vec{b}_* - \nu\triangle\vec{b}_* = \tag{41}$$

$$= \vec{b}_* \times \vec{B} - \nu\,curl\,\vec{B} + \nabla\,\Omega \ .$$

Let us notice that the forward and backward acceleration formulas *do not* coincide as was the case before (cf. Eq. (8), (12)). There is a definite time-asymmetry in the local description of the diffusion process in the presence of general magnetic fields, unless $curl\,\vec{B} = 0$. The quantity which is explicitly time-reversal invariant can be easily introduced:

$$\vec{v}(\vec{x}, t) = \frac{1}{2}(\vec{b} + \vec{b}_*)(\vec{x}, t) \Rightarrow \tag{42}$$

$$\frac{1}{2}(D_+^2 + D_-^2)(\vec{X}(t)) = \vec{v} \times \vec{B} + \nabla\,\Omega \ .$$

As yet there is no trace of Lorentzian electric forces, unless extracted from the term $\nabla\,\Omega(\vec{x}, t)$. This step we shall accomplish in Section 4.

For a probability density $\theta_*\theta = \rho$ of the related Markovian diffusion process, [34, 28], we would have fulfilled both the Fokker-Planck and the continuity equations: $\partial_t\rho = \nu\triangle\rho - \nabla(\vec{b}\rho) = -\nabla(\vec{v}\rho) = -\nu\triangle\rho - \nabla(\vec{b}_*\rho)$, as before (cf. Section 1).

In the above, the equation (41) can be regarded as the Burgers equation with a general external magnetic (plus other external force contributions if necessary) forcing, and its definition is an outcome of the underlying mathematical structure related to the adjoint pair (33), (37) of parabolic equations. Our construction shows that the solution of the magnetically forced Burgers equation needs to be sought in the form (39).

4 Schrödinger's interpolation in a constant magnetic field and quantally inspired generalisations

Presently, we shall confine our attention to the simplest case of a constant magnetic field, defined by the vector potential $\vec{A} = \{-\frac{B}{2}x_2, +\frac{B}{2}x_1, 0\}$. Here, $\vec{B} = \{0, 0, B\}$, $\nabla\vec{A} = 0$ and $curl\,\vec{B} = \vec{0}$ which significantly simplifies formulas (32)-(42).

As emphasized before, most of our discussion was based on the existence assumption for fundamental solutions of the (adjoint) parabolic equations (33), (37). For magnetic fields, which do not vanish at spatial infinities (hence for our "simplest" choice), the situation becomes rather complicated. Namely, an expression for

$$c_{\vec{A}}(\vec{x}, t) = c(\vec{x}, t) - \frac{B^2}{16\nu}(x_1^2 + x_2^2) \tag{43}$$

includes a *repulsive* harmonic oscillator contribution.

For the existence of a well defined Markovian diffusion process it appears necessary that a nonvanishing contribution from an unbounded from above $c(\vec{x}, t)$ would counterbalance the harmonic repulsion. To see that this *must be* the case, let us formally constrain $\theta(\vec{x}, t) = exp(\Phi(\vec{x}, t))$ to yield (in accordance with (9)) the identity:

$$c(\vec{x}, t) = \partial_t\Phi + \nu[\nabla\Phi]^2 + \nu\triangle\Phi = 0 \tag{44}$$

Then, we deal with the simplest version of the adjoint system (35), (37) where, in view of $\nabla\vec{A} = 0 = c$, there holds:

$$\partial_t\theta = -\nu[\nabla - \frac{1}{2\nu}\vec{A}]^2\theta = -\nu\triangle\theta + \vec{A}\nabla\theta - \frac{1}{4\nu}[\vec{A}]^2\theta \tag{45}$$

$$\partial_t\theta_* = \nu[\nabla + \frac{1}{2\nu}\vec{A}]^2\theta_* = \nu\triangle\theta_* + \vec{A}\nabla\theta_* + \frac{1}{4\nu}[\vec{A}]^2\theta_* \ .$$

With our choice, $curl\,\vec{A} = \{0, 0, B\}$, the equations (45) *do not* possess a fundamental solution, which would be well defined for *all* $(\vec{x}, t) \in R^3 \times R^+$: everything because of the harmonic repulsion term in the forward parabolic equation. In the Appendix, we shall prove that the function $k(\vec{y}, s, \vec{x}, t)$, :

$$k(\vec{y}, s, \vec{x}, t) = [\frac{B}{4\pi sin(\frac{1}{2}B(t-s))}](\frac{1}{2\pi(t-s)})^{1/2} \tag{46}$$

$$\times exp\{-\frac{1}{2(t-s)}(x_3 - y_3)^2 - \frac{B}{4}cot(\frac{B}{2}(t-s))[(x_2 - y_2)^2 + (x_1 - y_1)^2] - \frac{B}{2}(x_1 y_2 - x_2 y_1)\}$$

only when restricted to times $t - s \leq \frac{\pi}{B}$ is an acceptable example of a *unique* positive (actually, positivity extends to times $t - s \leq \frac{2\pi}{B}$) fundamental solution of the system (44), (rescaled to yield $\nu \to \frac{1}{2}$). Here, formally, (46) can be obtained from the expression (29) by the replacement $\vec{A} \to -i\vec{A}$.

An immediate insight into a harmonic repulsion obstacle can be achieved after an $x - y$-plane rotation of Cartesian coordinates: $x'_1 = x_1 cos(\omega t) - x_2 sin(\omega t)$, $x'_2 = x_1 sin(\omega t) + x_2 cos(\omega t)$, $x'_3 = x_3$, $t' = t$ with $\omega = \frac{B}{4\sqrt{\nu}}$. Then, equations (45) get transformed into an adjoint pair:

$$\partial_{t'}\theta = -\nu\triangle'\theta - \omega^2(x'^2_1 + x'^2_2)\theta \tag{47}$$

$$\partial_{t'}\theta_* = \nu\triangle'\theta_* + \omega^2(x'^2_1 + x'^2_2)\theta_* \ .$$

Notice that the transformation $\omega \to i\omega$ would replace repulsion in (47) by the harmonic attraction. On the other hand, we can get rid of the repulsive term by assuming that $c(\vec{x}, t)$, (43) does not identically vanish. For example, we can formally demand that, instead of (44), $c(\vec{x}, t) = +\frac{B^2}{8\nu}(x_1^2 + x_2^2)$ plays the rôle of an electric potential. Then, harmonic attraction replaces repulsion in the final form of equations (35), (37).

As a byproduct, we are given a transition probability density of the diffusion process governed by the adjoint system (cf. (28)):

$$\partial_t\theta = -\nu\triangle\theta + \vec{A}\nabla\theta \tag{48}$$

$$\partial_t\theta_* = \nu\triangle\theta_* + \vec{A}\nabla\theta_* \ .$$

with $\vec{A} = \frac{B}{2}\{-x_2, x_1, 0\}$. Namely, by means of the previous $x - y$-plane rotation, (48) is transformed into a pair of time adjoint heat equations:

$$\partial_{t'}\theta = -\nu\triangle'\theta \ , \ \partial_{t'}\theta_* = \nu\triangle'\theta_* \tag{49}$$

whose fundamental solution is the standard heat kernel.

Finding explicit analytic solutions of rather involved equations (35), (37) is a formidable task on its own, in contrast to much simpler-unforced or conservatively forced dynamics issue.

Interestingly, we can produce a number of examples by invoking the quantum Schrödinger dynamics. This quantum inspiration has been proved to be very useful in the past, [34, 35]. At this point, we shall follow the idea of Ref. [28] where the strategy developed for solving the Schrödinger boundary data problem has been applied to quantally induced stochastic processes (e.g. Nelson's diffusions, [33, 36]). They were considered as a particular case of the general theory appropriate for nonequilibrium statistical physics processes as governed by the adjoint pair (18), and exclusively in conjunction with Born's statistical postulate in quantum theory.

The Schrödinger picture quantum evolution is then consistently representable as a Markovian diffusion process. All that follows from the previously outlined Feynman-Kac kernel route, [33, 36, 34, 28, 29, 39, 40], based on exploiting the adjoint pairs of parabolic equations. However, the respective semigroup theory has been developed for pure gradient drift fields, hence without reference to any impact of electromagnetism on the pertinent diffusion process. And electromagnetism is definitely ubiquitous in the world of quantum phenomena.

Let us start from an ordinary Schrödinger equation for a charged particle in an arbitrary external electromagnetic field, in its standard dimensional form. To conform with the previous notation let us absorb the charge e and mass m parameters in the definition of $\vec{A}(\vec{x})$ and the potential $\phi(\vec{x})$, e.g. we consider B instead of $\frac{e}{m}B$ and ϕ instead ϕ/m. Additionally, we set ν instead of $\frac{\hbar}{2m}$. Then, we have:

$$i\partial_t \psi(\vec{x}, t) = -\nu(\nabla - \frac{i}{2\nu}\vec{A})^2 \psi(\vec{x}, t) + \frac{1}{2\nu}\phi(\vec{x})\psi(\vec{x}, t) \ . \tag{50}$$

The standard Madelung substitution $\psi = exp(R + iS)$ allows to introduce the real functions $\theta = exp(R + S)$ and $\theta_* = exp(R - S)$ instead of complex ones $\psi, \overline{\psi}$. They are solutions of an adjoint parabolic system (35), (37), where the impact of (50) is encoded in a specific functional form of, otherwise arbitrary potential $c(\vec{x}, t)$:

$$c(\vec{x}, t) = \frac{1}{2\nu}\Omega(\vec{x}, t) = \frac{1}{2\nu}[2Q(\vec{x}, t) - \phi(\vec{x})] \tag{51}$$

$$Q(\vec{x}, t) = 2\nu^2 \frac{\triangle\rho^{1/2}(\vec{x}, t)}{\rho^{1/2}(\vec{x}, t)} = 2\nu^2[\triangle R(\vec{x}, t) + [\nabla R(\vec{x}, t)]^2] \ .$$

The quantum probability density $\rho(\vec{x}, t) = \psi(\vec{x}, t)\overline{\psi}(\vec{x}, t) = \theta(\vec{x}, t)\theta_*(\vec{x}, t)$ displays a factorisation $\rho = \theta\theta_*$ in terms of solutions of adjoint parabolic equations, which we recognize to be characteristic for probabilistic solutions (Markov diffusion processes)

of the Schrödinger boundary data problem (cf. Section 1), [34, 29, 28, 39]. It is easy to verify the validity of the Fokker-Planck equation whose forward drift has the form (38). Also, equations (40), (41) do follow with $\Omega = 2Q - \phi$.

By defining $\vec{E} = -\nabla\phi$ (with ϕ utilised instead of $\frac{e}{m}\phi$), we immediately arrive at the complete Lorentz force contribution in all acceleration formulas (before, we have used $curl\,\vec{B} = 0$):

$$\partial_t\vec{b} + (\vec{b}\nabla)\vec{b} + \nu\triangle\vec{b} = \vec{b}\times\vec{B} + \vec{E} + \nu curl\,\vec{B} + 2\nabla Q \tag{52}$$

$$\partial_t\vec{b}_* + (\vec{b}_*\nabla)\vec{b}_* - \nu\triangle\vec{b}_* = \vec{b}_*\times\vec{B} + \vec{E} - \nu curl\,\vec{B} + 2\nabla Q$$

Moreover, the velocity field named the current velocity of the flow, $\vec{v} = \frac{1}{2}(\vec{b} + \vec{b}_*)$, enters the familiar local conservation laws (see also [29] for a discussion of how the "quantum potential" Q affects such laws in case of the standard Brownian motion and Smoluchowski-type diffusion processes)

$$\partial\rho = -\nabla(\vec{v}\rho) \tag{53}$$

$$\partial_t\vec{v} + (\vec{v}\nabla)\vec{v} = \vec{v}\times\vec{B} + \vec{E} + \nabla Q \ .$$

A comparison with (33)-(43) shows that equations (50)-(53) can be regarded as the specialized version of the general external forcing problem with an explicit electromagnetic (Lorentz force-inducing) contribution and an arbitrary term of non-electromagnetic origin, which we denote by $c(\vec{x}, t)$ again. Obviously, c is represented in (51), by $\frac{1}{\nu}Q(\vec{x}, t)$.

We have therefore arrived at the following ultimate generalization of the adjoint parabolic system (18), that encompasses the nonequilibrium statistical physics and essentially quantum evolutions on an equal footing (with no clear-cut discrimination between these options, as in Ref. [28]) and gives rise to an external (Lorentz) electromagnetic forcing:

$$\partial_t\theta(\vec{x}, t) = [-\nu(\nabla - \frac{1}{2\nu}\vec{A})^2 - \frac{1}{2\nu}\phi(\vec{x}) + c(\vec{x}, t)]\theta(\vec{x}, t) \tag{54}$$

$$\partial_t\theta_*(\vec{x}, t) = [\nu(\nabla + \frac{1}{2\nu}\vec{A})^2 + \frac{1}{2\nu}\phi(\vec{x}) - c(\vec{x}, t)]\theta_*(\vec{x}, t) \ .$$

A subsequent generalisation encompassing time-dependent electromagnetic fields is immediate.

The adjoint parabolic pair (54) of equations can thus be regarded to determine a Markovian diffusion process in exactly the same way as (18) did. If only a suitable choice of vector and scalar potentials in (54) guarrantees a continuity and positivity

of the involved semigroup kernel (take the Radon-Nikodym density of the form (34), with $\Omega \to -\phi + \Omega$, and integrate with respect to the conditional Wiener measure), then the mere knowledge of such integral kernel suffices for the implementation of steps (18)-(22), with $u \to \theta_*$, $v \to \theta$. To this end it is not at all necessary that $k(\vec{x}, s, \vec{y}, t)$ is a fundamental solution of (54). A sufficient condition is that the semigroup kernel is a continuous (and positive) function. The kernel may not even be differentiable, see e.g. Ref. [28] for a discussion of that issue which is typical for quantal situations.

After adopting (54) as the principal dynamical ingredient of the electromagnetically forced Schrödinger interpolation, we must slightly adjust the emerging acceleration formulas. Namely, they have the form (52) , but we need to replace $2Q(\vec{x}, t)$ by, from now on arbitrary, potential $\Omega(\vec{x}, t) = 2\nu c(\vec{x}, t)$. The second equation in (53) also takes a new form:

$$\partial_t \vec{v} + (\vec{v}\nabla)\vec{v} = \vec{v} \times \vec{B} + \vec{E} + \nabla(\Omega - Q) \tag{55}$$

see e.g. Ref. [29] for more detailed explanation of this step. The presence in (54) of the density-dependent $-\nabla Q$ term finds its origin in the identity $\vec{b} - \vec{b}_* = 2\nu\nabla\rho(\vec{x}, t)$ and is a necessary consequence of the involved (forced in the present case) Brownian motion, see e.g. [64, 65, 38].

Finally, the second equation (52) with Ω replacing $2Q$ is the most general form of the Burgers equation with an external forcing, where the electromagnetic (Lorentz force) contribution has been extracted for convenience. Solutions of this equation must be sought for in the form (39), which generalizes the logarithmic Hopf-Cole transformation to non-gradient drift fields. Equations (54) are the associated parabolic partial differential (generalised heat) equations, that completely determine probabilistic solutions (Markovian diffusion processes) of the Schrödinger boundary data (interpolation) problem, for which the forced Burgers velocity fields play the rôle of backward drifts of the process.

Appendix

General criterions, allowing to decide when the semigroup kernel associated with the adjoint pair (54) is a continuous and positive function, belong to a specialised branch of mathematical theory of parabolic partial differential equations and mathematical physics. A list of restrictions suitable for our case can be found in Section 4 of Ref. [63].

To give a flavour of a typical argumentation, we shall present a more selective discussion of conditions under which the quantally inspired parabolic problem (33),

(37), (51) admits fundamental solutions. Let us point out that quite generally fundamental solutions do not exist (the kernels do !) in such case, cf. [28, 40].

Let us rescale the problem (50) so that we are interested in:

$$i\partial_t \psi(\vec{x}, t) = -\frac{1}{2}(\nabla - i\vec{A})^2 \psi(\vec{x}, t) + \phi(\vec{x})\psi(\vec{x}, t) \ . \tag{A.1}$$

with the initial data $\psi_0(\vec{x}) \in L^2(R^3)$. We assume that $A_i, i = 1, 2, 3, \in C^1(R^3)$ i.e. have continuous first derivatives, while $\phi(\vec{x}) \in C(R^3)$ and is bounded from below, e.g. $\phi(\vec{x}) > -M$ for some positive M. It is well known that under these restrictions the operator $H = \frac{1}{2}(i\nabla + \vec{A})^2 + \phi$ is essentially selfadjoint on $C_c^\infty(R^3)$, i.e. the space of smooth functions with a compact support, [66]. Thus $\psi_t = exp(-it\overline{H})\psi_0$ is a solution of (A.1) for every time t and any initial condition $\psi_0 \in D(\overline{H})$. Here, $\overline{H}$ denotes the closure of an operator H.

Let us fix some $T > 0$ and let us assume that $\psi(\vec{x}, t) \neq 0$ for all $\vec{x} \in R^3$ and $t \in [0, T]$. Then, the Madelung substitution $\psi = exp(R + iS)$ implies:

$$\partial_t R = -\nabla R \nabla S - \frac{1}{2}\triangle S + \frac{1}{2}\nabla \vec{A} + \vec{A}\nabla R \tag{A.2}$$

$$\partial_t S = \frac{1}{2}(\nabla R)^2 - \frac{1}{2}(\nabla S)^2 + \frac{1}{2}\triangle R + \vec{A}\nabla S - \frac{1}{2}\vec{A}^2 - \phi \ .$$

At the same time, the real-valued functions $\theta = exp(R + S)$ and $\theta_* = exp(R - S)$ obey the adjoint parabolic equations (to be compared with (35), (37)):

$$\partial_t \theta_* = \frac{1}{2}\triangle\theta_* + \vec{A}\nabla\theta_* - c_{\vec{A},\phi}\theta_* \tag{A.3}$$

$$\partial_t \theta = -\frac{1}{2}\triangle\theta + \nabla(\vec{A}\theta) + c_{\vec{A},\phi}\theta$$

where

$$c_{\vec{A},\phi} = \frac{1}{2}\nabla\vec{A} - \frac{1}{2}\vec{A}^2 + 2Q - \phi \tag{A.4}$$

$$2Q = \frac{\triangle\rho^{1/2}}{\rho^{1/2}} = (\nabla R)^2 + \triangle R \ .$$

Let us furthermore assume that $\partial A_i/\partial x_j$ and $c_{\vec{A},\phi}$ are locally Hölder continuous and moreover that

$$|A_i| \leq K_1(\vec{x}^2 + 1)^{1/2} \tag{A.5}$$

$$|c_{\vec{A},\phi}| < K_2(\vec{x}^2 + 1)$$

for all $i = 1, 2, 3$. Then, for a certain T_0 which may be smaller that the previously chosen T, there exists a unique fundamental solution $k(\vec{y}, s, \vec{x}, t)$ defined for

all $\vec{x}, \vec{y} \in R^3$, *but* confined to a time interval $0 \le s < t \le T_0$.

The solution has the following properties:
(i) For any $\eta \in C_c(R^3)$, $\theta_*(\vec{x}, t) = \int_{R^3} k(\vec{y}, s, \vec{x}, t)\eta(\vec{y})d^3y$ is a regular solution of the forward equation (A.3) such that $lim_{t \downarrow s}\theta_*(\vec{x}, t) = \phi(\vec{x})$.
(ii) k is a regular solutions of the forward equation (A.3) in $(\vec{x}, t)$ variables and a regular solution of the backward equation (A.3) in the $(\vec{y}, s)$ variables.
(iii) $\int_{R^3} k(\vec{y}, s, \vec{x}, t)d^3y \le C_1 exp(C_2 x^2)$ for some positive C_1 and C_2.
(iv) k is strictly positive i.e. $k > 0$.
(v) k obeys the Chapman-Kolmogorov equation $\int_{R^3} k(\vec{y}, s, \vec{z}, u)k(\vec{z}, u, \vec{x}, t)d^3z = k(\vec{y}, s, \vec{x}, t)$ for all $s < u < t \le T_0$.

In the above, properties (i) and the first part of (ii) are in fact a definition of a fundamental solution, [22]. The second property (ii) and the estimate (iii) are canonical properties of the fundamental solution,[60, 62]. The property (iv) comes as a consequence of another estimate (Lemma 2 in Ref. [62]): for every $\epsilon \in (0, t)$ and every $x \in R^3$ there exist constants Λ and μ such that $k(\vec{y}, s, \vec{x}, t) \ge \Lambda exp[-\mu(\vec{x}-\vec{y})^2]$ for all $\vec{y} \in R^3$ and all $s \in [0, t - \epsilon]$. The Chapman-Kolmogorov equation (property (v)) follows from the uniqueness of positive solutions of the forward equation (A.3), [62, 60].
Consequently, what needs to be proven is the *uniqueness* of the integral kernel k.
To this end, let us assume that there are two fundamental solutions. Then, for any $\eta \in C_c(R^3)$ we have

$$u_i(\vec{x}, t) = \int_{R^3} k_i(\vec{y}, s, \vec{x}, t)\eta(\vec{y})d^3y \qquad (A.6)$$

for $i = 1, 2$, and $u_i(\vec{x}, t)$ are the regular solutions such that $u(\vec{x}, 0) = \eta(\vec{x})$.
Let us define $u = u_2 - u_1$. Then, for $k_0 > C_2$ there holds:

$$\int_0^{T_0} \int_{R^3} |u(\vec{x}, t)|exp(-k_0\vec{x}^2)d^3xdt \le \qquad (A.7)$$

$$\le \int_0^{T_0} \int_{R^3} \int_{R^3} (k_1 + k_2)(\vec{y}, s, \vec{x}, t)\eta(\vec{y})exp(-k_0\vec{x}^2)d^3yd^3xdt \ .$$

But we have $\int_{R^3} k_i(\vec{y}, s, \vec{x}, t)d^3y \le C_1 exp(C_2\vec{x}^2)$ for $i = 1, 2$. Consequently:

$$\int_0^{T_0} \int_{R^3} |u(\vec{x}, t)|exp(-k_0x^2)d^3xdt \le 2||\eta||_{sup} C_1 T_0 \int_{R^3} exp[-(k_0 - C_2)\vec{x}^2]d^3x < \infty \ . \qquad (A.8)$$

By Theorem AB2 of Ref. [62], in view of $u(\vec{x}, 0) = 0$ we here arrive at $u(\vec{x}, t) \equiv 0$. Since η was arbitrary, we conclude that $k_1 \equiv k_2$. The uniqueness has been proved.

One may argue about the seemingly spurious time interval restriction to $T_0 < T$. This time domain limitation for the kernel k can be easily justified. Namely, let us consider the forward equation (cf. A.3) $\partial_t u = \frac{1}{2}\triangle u + \vec{A}\nabla u + cu$ and define $v = u\,exp[-g(\vec{x},t)]$ where $g(\vec{x},t) = \frac{\vec{x}^2}{1-at}$ and $a > 0$. The newly introduced function v satisfies:

$$\partial_t v = \frac{1}{2}\triangle v + (\vec{B}\nabla)v + c_0 v \qquad (A.9)$$

with $\vec{B} = \vec{A} + \frac{1}{2}\nabla g$ and $c_0 = c_{\vec{A},\phi} + \frac{1}{2}(\nabla g)^2 + \frac{1}{2}\triangle g + (\vec{A}\nabla)g - \partial_t g$. Because of $\partial_t g = \frac{a\vec{x}^2}{(1-at)^2}$, by taking a large enough we can make the equation (A.9) "dissipative", [67], and so construct its fundamental solution k_0. A fundamental solution of the forward equation (A.3) is the given by:

$$k(\vec{y},s,\vec{x},t) = k_0(\vec{y},s,\vec{x},t)exp[g(\vec{x},t) - g(\vec{y},s)] \ . \qquad (A.10)$$

However if a is large, then t can run only a small interval $[0, 1/a]$.

All the above features of the fundamental solution, (i)-(v), can be explicitly checked (by inspection) for the kernel (46), associated with the parabolic pair (45) in a constant magnetic field. They hold true for times $0 \le s < t \le T_0 < \pi$, except for the property (iii) which is true *only* if $T < \frac{\pi}{2}$. Indeed:

$$\frac{1}{2\pi sin(t-s)}\int_{R^2} exp\{-\frac{cot(t-s)}{2}[(x_1 - y_1)^2 + (x_2 - y_2)^2]\}\, exp(-x_2 y_1 + x_1 y_2)dy_1 dy_2$$
$$(A.11)$$
$$= cos(t-s)\, exp[\frac{tan(t-s)}{2}(x_1^2 + x_2^2)] \ .$$

Since $[2\pi(t-s)]^{-1/2}\int_R exp[-(x_3 - y_3)^2/2(t-s)]dy_3 = 1$, to achieve (iii) it suffices to put $C_1 = 1$ and $C_2 = \frac{1}{2}tan T_0$, provided $T_0 < \frac{\pi}{2}$.

The uniqueness of the kernel for time $T_0 < \frac{\pi}{2}$ is proved along the same lines as this for the fundamental solution of (A.3).

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